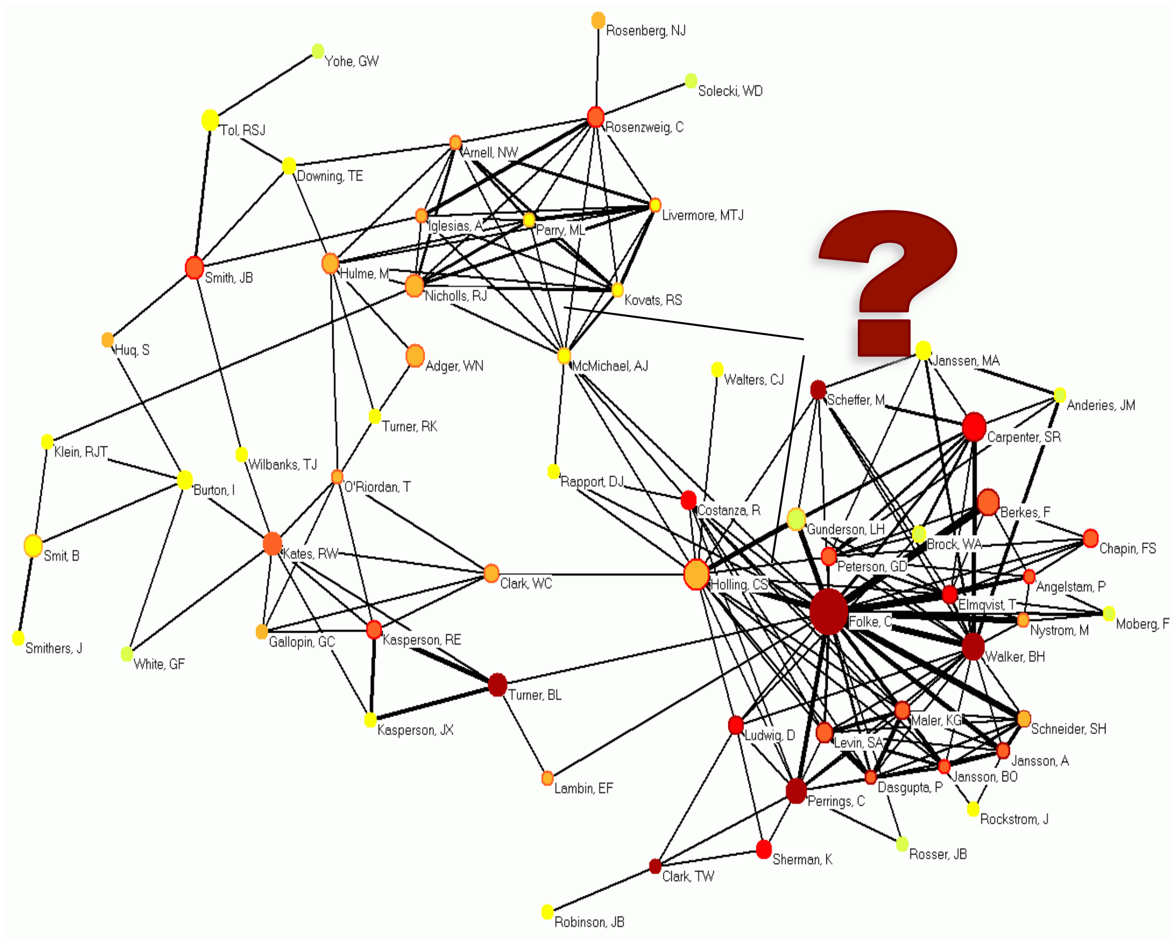
Paths and
Random Walks on Graphs

Based on materials by Lala Adamic and
Purnamrita Sarkar

Motivation: Link prediction in social networks



Motivation: Basis for recommendation

Motivation: Personalized search

Web Images Video News Maps Gmail more ▾

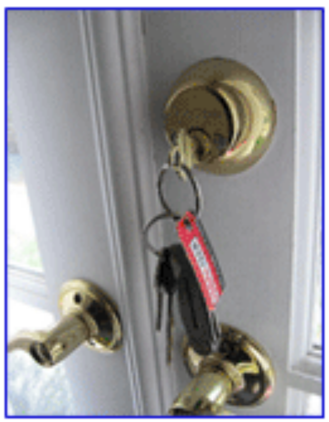
Google

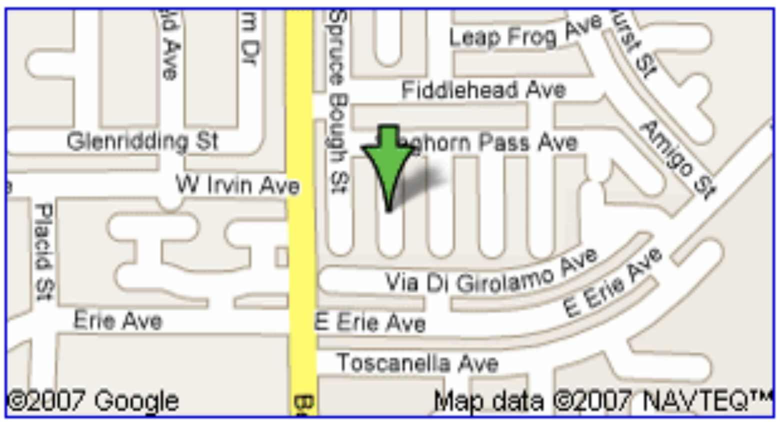
my car keys

Search

Advanced Search
Preferences

Web





In the front door, where you left them last night.

Where Are **My Car Keys**?

Why graphs?

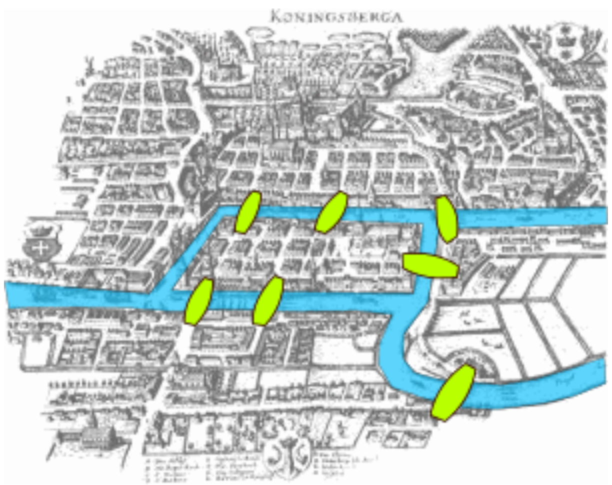
- The underlying data is naturally a graph
  - Papers linked by citation
  - Authors linked by co-authorship
  - Bipartite graph of customers and products
  - Web-graph
  - Friendship networks: who knows whom

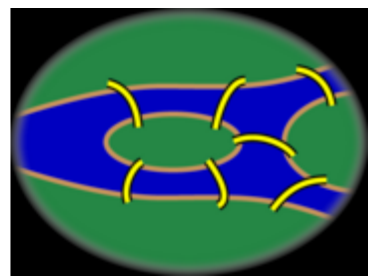
What are we looking for

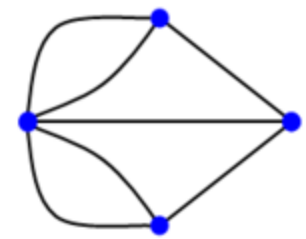
- Rank nodes for a particular query
  - Top k matches for "Random Walks" from Citeseer
  - Who are the most likely co-authors of "Manuel Blum".
  - Top k book recommendations for Jen from Amazon
  - Top k websites matching "Sound of Music"
  - Top k friend recommendations for Bob when he joins "Facebook"

History: Graph theory

- Euler's **Seven Bridges of Königsberg** – one of the first problems in graph theory
- Is there a route that crosses each bridge only once and returns to the starting point?



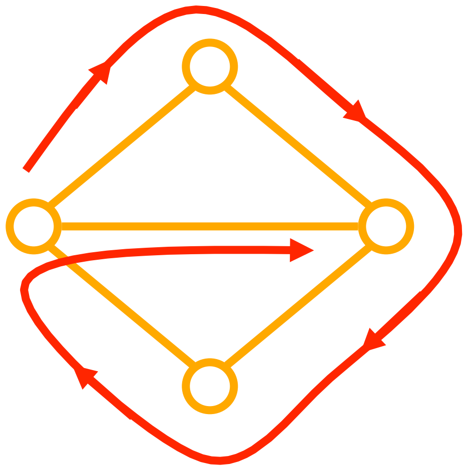




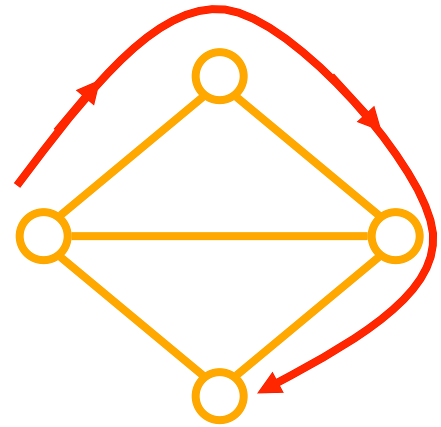
Source: http://en.wikipedia.org/wiki/Seven_Bridges_of_Königsberg

Eulerian paths

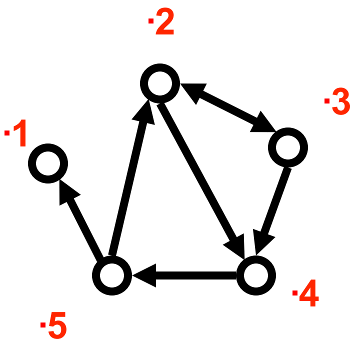
- If starting point and end point are the same:
  - only possible if no nodes have an odd degree
    - each path must visit and leave each shore
- If don’t need to return to starting point
  - can have 0 or 2 nodes with an odd degree



·Eulerian path: traverse each
·edge exactly once



·Hamiltonian path: visit
·each vertex exactly once



Node degree from matrix values

- Outdegree = $\sum_{j=1}^{n} A_{ij}$

·A = $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$

·example: outdegree for node 3 is 2, which we obtain by summing the number of non-zero entries in the 3rd *row*

$$\sum_{j=1}^{n} A_{3j}$$

- Indegree = $\sum_{i=1}^{n} A_{ij}$

·A = $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$

·example: the indegree for node 3 is 1, which we obtain by summing the number of non-zero entries in the 3rd *column*

$$\sum_{i=1}^{n} A_{i3}$$

Definitions

- n x n Adjacency matrix A.
  - A(i,j) = weight on edge from *i* to *j*
  - If the graph is undirected A(i,j)=A(j,i), i.e. A is symmetric

- n x n Transition matrix P.
  - P is row stochastic
  - P(i,j) = probability of stepping on node j from node i

    $= A(i,j)/\sum_i A(i,j)$

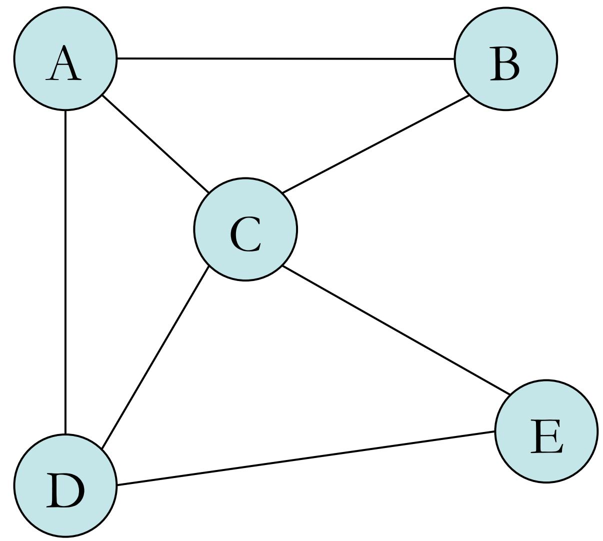
- n x n Laplacian Matrix L.
  - $L(i,j)=\sum_i A(i,j)-A(i,j)$
  - Symmetric positive semi-definite for undirected graphs
  - Singular

Definitions

Adjacency Matrix

A=

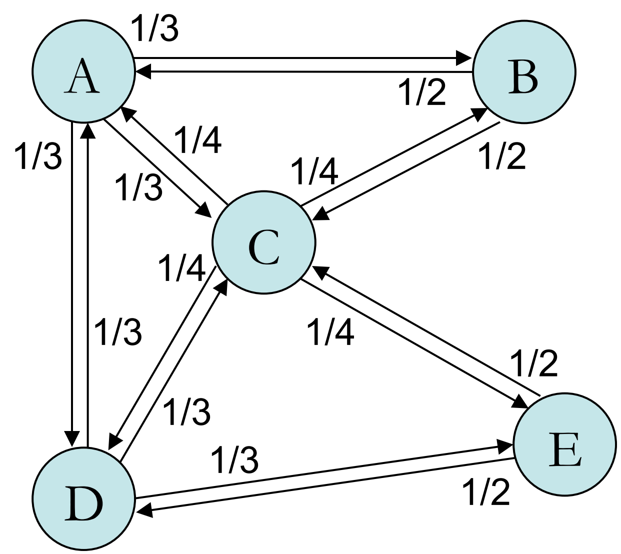
	1	1	1	
1		1		
1	1		1	1
1		1		1
		1	1	



Transition Matrix

P=

	1/3	1/3	1/3	
1/2		1/2		
1/4	1/4		1/4	1/4
1/3		1/3		1/3
		1/2	1/2	

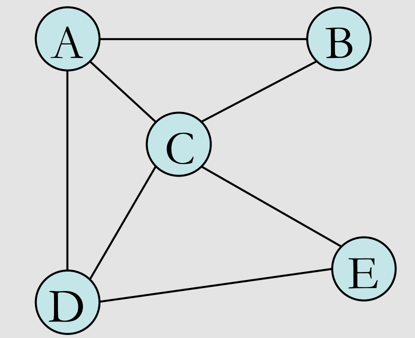


Definitions

Graph Laplacian

$$L = D - A \qquad D = \text{diag}(d)$$

$$\begin{array}{c|ccccc} & A & B & C & D & E \\ \hline A & & & & & \\ B & & & & & \\ C & & & & & \\ D & & & & & \\ E & & & & & \end{array} = \begin{bmatrix} 3 & & & & \\ & 2 & & & \\ & & 4 & & \\ & & & 3 & \\ & & & & 2 \end{bmatrix} - \begin{bmatrix} & 1 & 1 & 1 & \\ 1 & & 1 & & \\ 1 & 1 & & 1 & 1 \\ 1 & & 1 & & 1 \\ & & 1 & 1 & \end{bmatrix}$$



$$\mathbf{A} = \begin{bmatrix} & 1 & 1 & 1 & \\ 1 & & 1 & & \\ 1 & 1 & & 1 & 1 \\ 1 & & 1 & & 1 \\ & & 1 & 1 & \end{bmatrix}$$

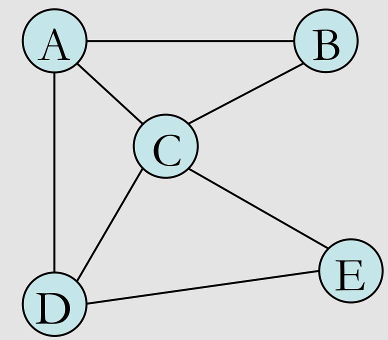
$$\mathbf{d} = \begin{bmatrix} 3 & 2 & 4 & 3 & 2 \end{bmatrix}$$

Spectral Graph Analysis

Graph Laplacian

$$L = D - A \qquad D = \mathrm{diag}(d)$$

$$\begin{array}{c|ccccc} & A & B & C & D & E \\ \hline A & & & & & \\ B & & & & & \\ C & & & & & \\ D & & & & & \\ E & & & & & \end{array} \; = \begin{bmatrix} 3 & & & & \\ & 2 & & & \\ & & 4 & & \\ & & & 3 & \\ & & & & 2 \end{bmatrix} - \begin{bmatrix} & 1 & 1 & 1 & \\ 1 & & 1 & & \\ 1 & 1 & & 1 & 1 \\ 1 & & 1 & & 1 \\ & & 1 & 1 & \end{bmatrix}$$



$$\mathbf{A} = \begin{bmatrix} & 1 & 1 & 1 & \\ 1 & & 1 & & \\ 1 & 1 & & 1 & 1 \\ 1 & & 1 & & 1 \\ & & 1 & 1 & \end{bmatrix}$$

$$\mathbf{d} = \begin{bmatrix} 3 & 2 & 4 & 3 & 2 \end{bmatrix}$$

Take the *eigendecomposition* of L

$$L = Q \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \lambda_3 & & \\ & & & \lambda_4 & \\ & & & & \lambda_5 \end{bmatrix} Q^{\mathsf{T}}$$

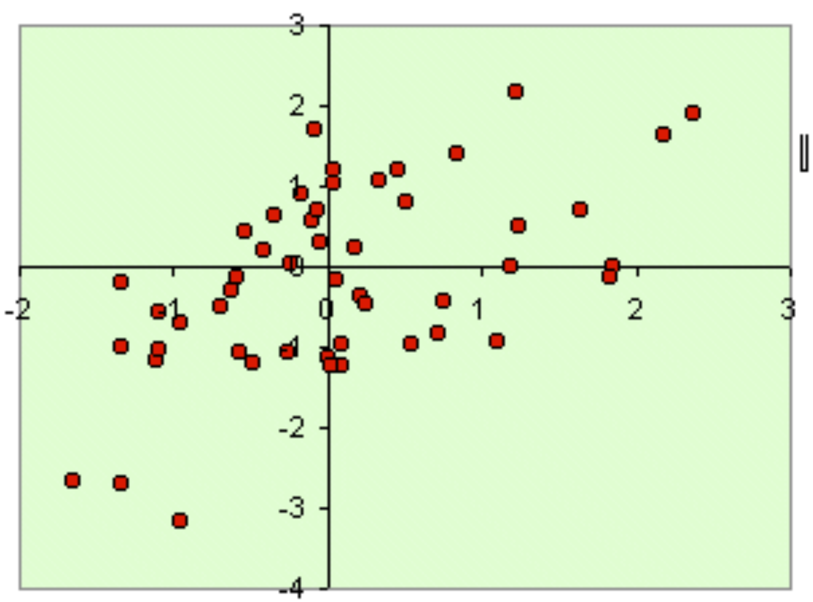
Eigenvectors

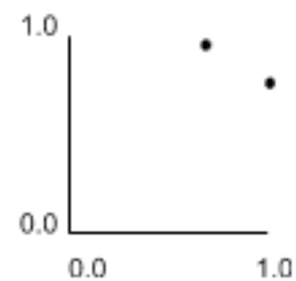
- Intuitive definition: An eigenvector is a direction for a matrix
- An eigenvector of an *n x n* matrix A is a vector such that $Av = \lambda v$, where *v* is the eigenvector and λ is the corresponding eigenvalue
  - Multiplying vector v by the scalar λ effectively stretches or shrinks the vector
- An *n x n* matrix should have *n* linearly independent eigenvectors

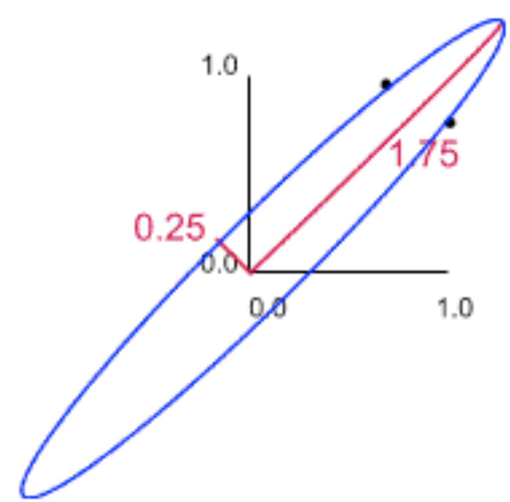
Eigenvectors Illustrated

- Consider an elliptical data cloud. The eigenvectors are then the major and minor axes of the ellipse

1.00	0.75
0.75	1.00







Spectral Graph Analysis

$$
\underbrace{L}_{} = \overset{Q}{\begin{bmatrix} q_1 & q_2 & q_3 & q_4 & q_5 \end{bmatrix}} \; \overset{\Lambda}{\begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \lambda_3 & & \\ & & & \lambda_4 & \\ & & & & \lambda_5 \end{bmatrix}} \; \overset{Q^T}{\begin{bmatrix} q_1^T \\ q_2^T \\ q_3^T \\ q_4^T \\ q_5^T \end{bmatrix}}
$$

Eigenvector q_1 is constant

Eigenvalue $\lambda_1 = 0$

L

3	-1	-1	-1	
-1	2	-1		
-1	-1	4	-1	-1
-1		-1	3	-1
		-1	-1	2

Q

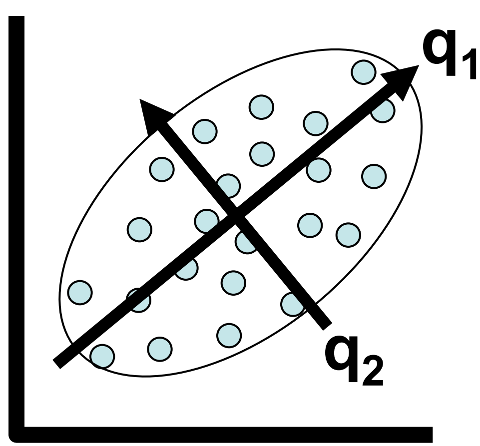
	q_1	q_2	q_3	q_4	q_5
A	0.45	-0.27	-0.5	-0.65	0.22
B	0.45	-0.65	0.5	0.27	0.22
C	0.45	-0.00	0.00	0.00	-0.89
D	0.45	0.27	-0.5	0.65	0.22
E	0.45	0.65	0.5	-0.27	0.22

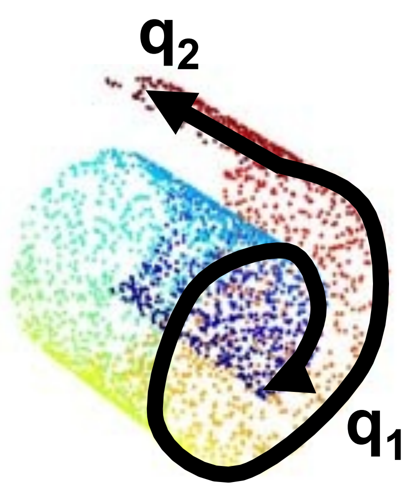
Λ

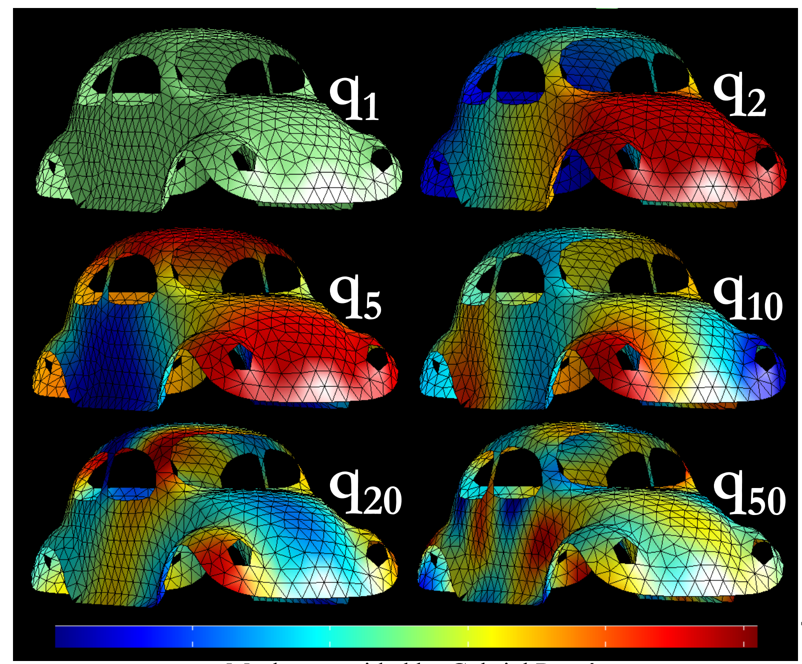
1	2	3	4	5
0.00	0	0	0	0
0	1.59	0	0	0
0	0	3.00	0	0
0	0	0	4.41	0
0	0	0	0	5.00

Spectral Graph Analysis

$$L = \underbrace{\begin{bmatrix} q_1 & q_2 & q_3 & q_4 & q_5 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \lambda_3 & & \\ & & & \lambda_4 & \\ & & & & \lambda_5 \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} q_1^T \\ q_2^T \\ q_3^T \\ q_4^T \\ q_5^T \end{bmatrix}}_{Q^T}$$







Meshes provided by Gabriel Peyré

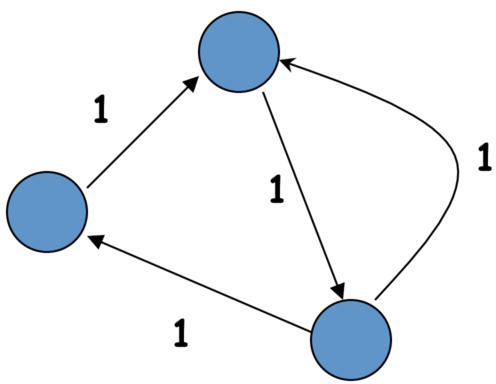
Random Walks

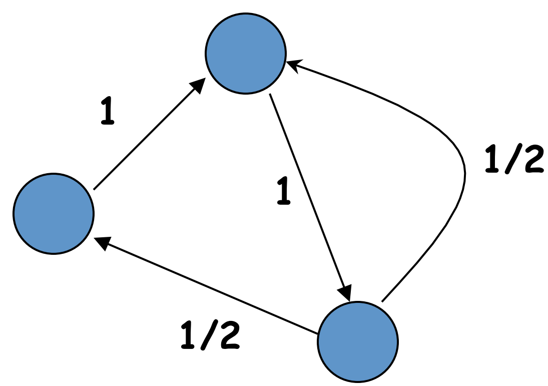
$$\begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{matrix}$$

Adjacency matrix A

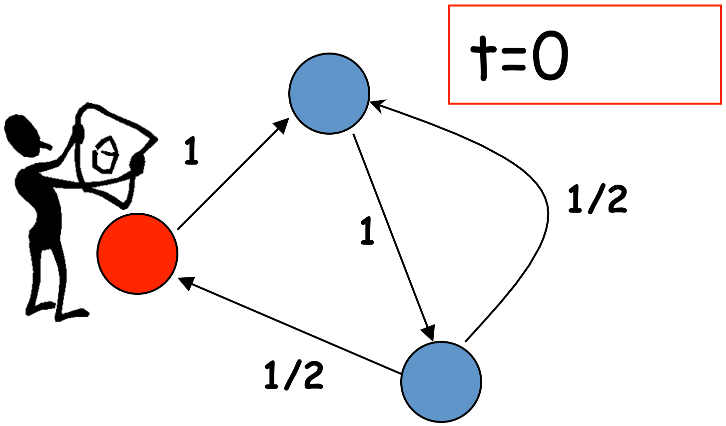
$$\begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{matrix}$$

Transition matrix P





What is a random walk



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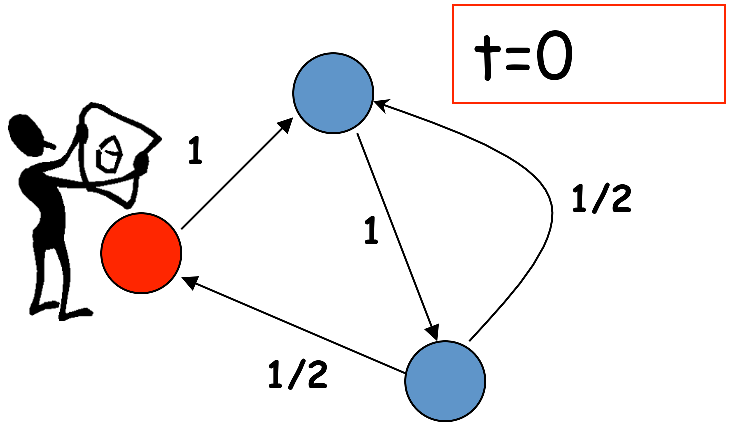

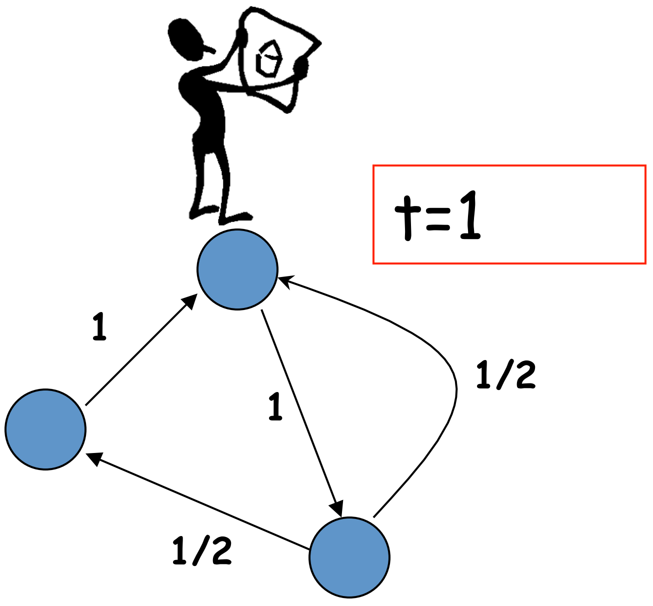

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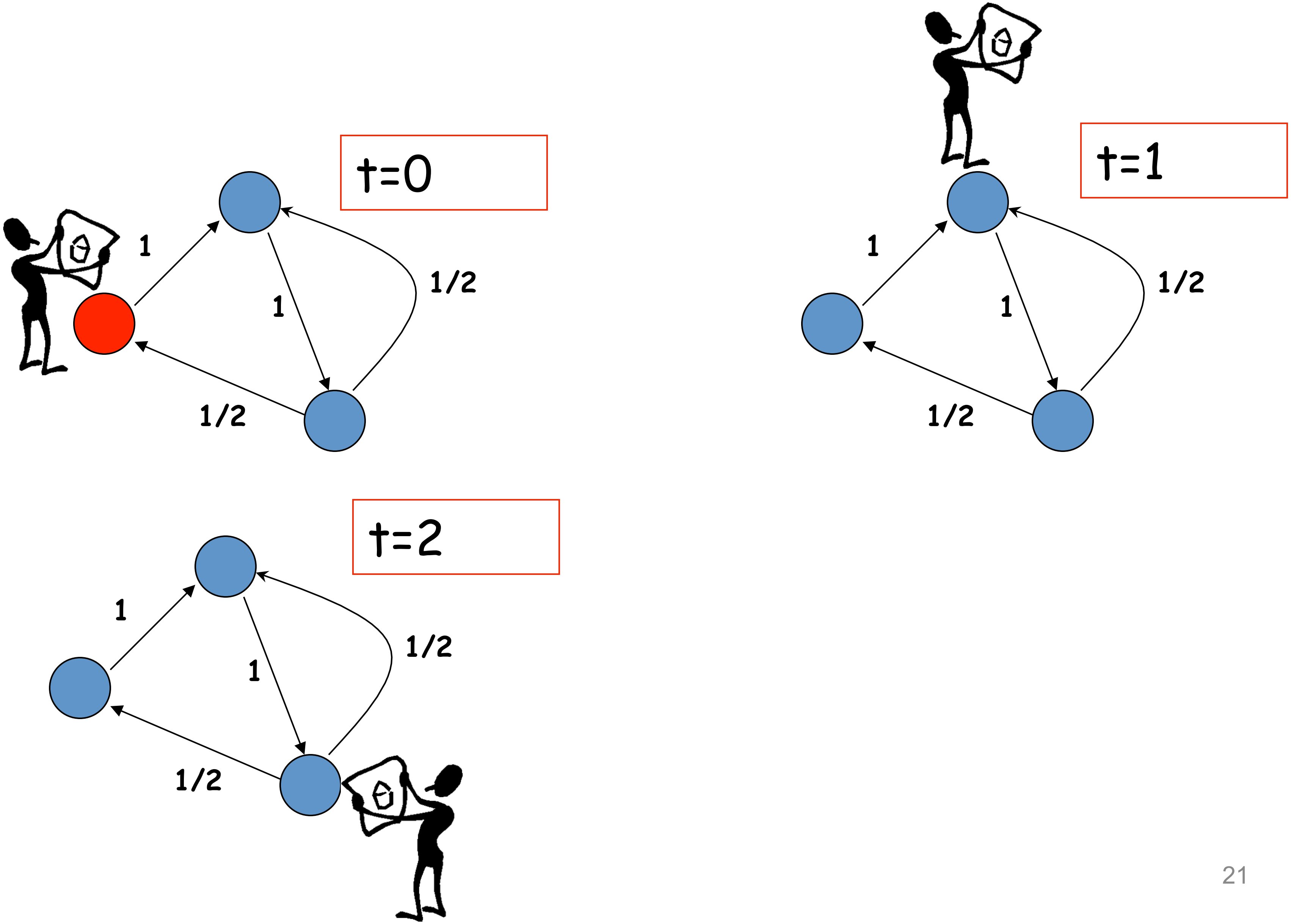

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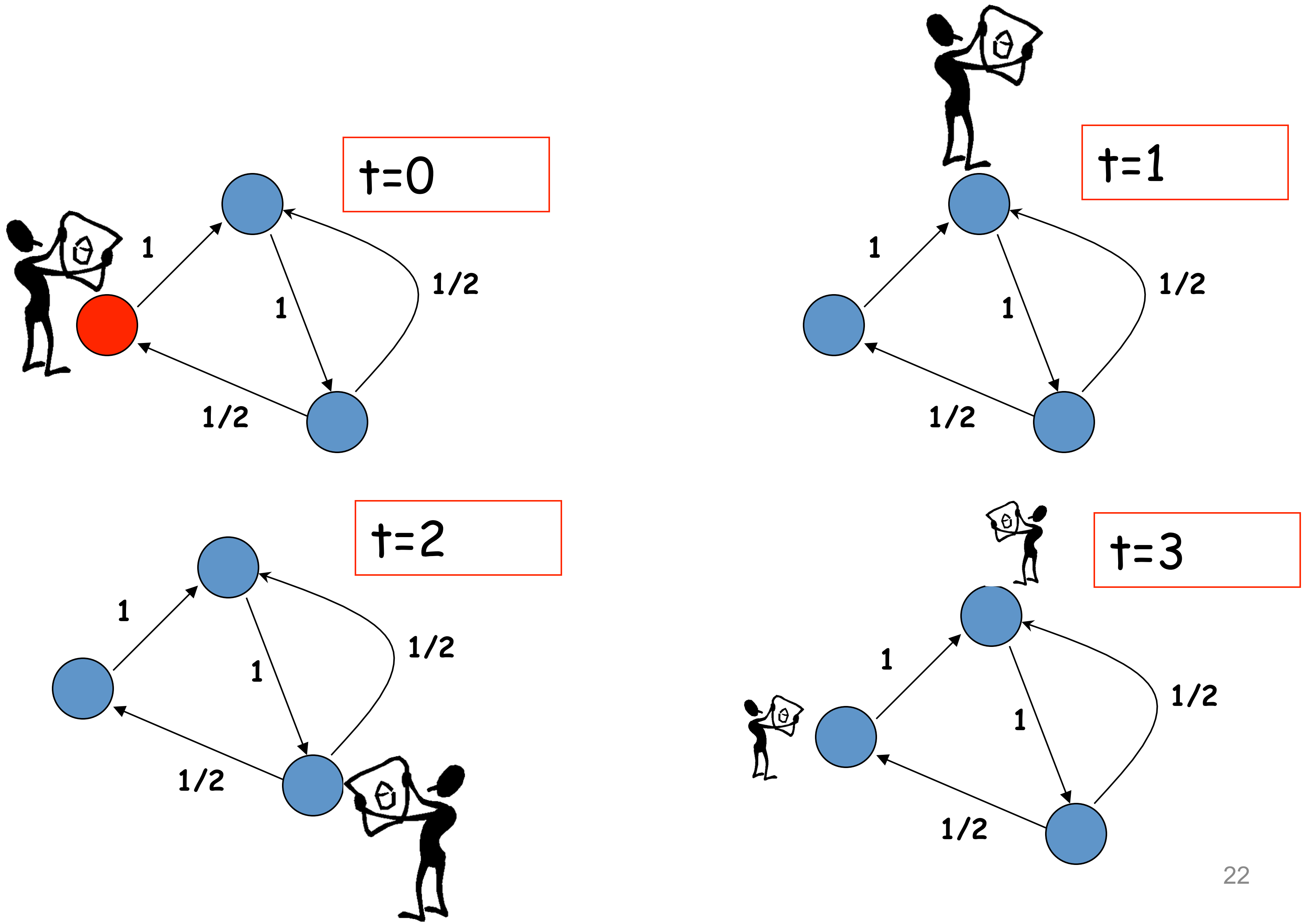

Probability Distributions

- $\phi_i^{(t)}$ = probability that the surfer is at node *i* at time *t*
- $$\phi_i^{(t+1)} = \sum_j \phi_i^{(t)} \times \Pr(j \to i)$$
- $$\begin{aligned} \phi_i^{(t+1)} &= \phi_i^{(t)} \times P \\ &= \phi_i^{(t-1)} \times P \times P \\ &= \phi_i^{(t-2)} \times P \times P \times P \\ &\ldots \\ &= \phi_i^{(0)} \times P^t \end{aligned}$$
- What happens when the surfer walks for a long time?

Stationary Distribution

- When the surfer keeps walking for a long time
- When the distribution does not change anymore
  - i.e. $\phi^{(t+1)} = \phi^{(t)}$
- For "well-behaved" graphs this does not depend on the start distribution!!

What is a stationary distribution? Intuitively and Mathematically

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- The stationary distribution at a node is related to the amount of time a random walker spends visiting that node.

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- For the stationary distribution $\phi^{(\infty)}$ we have

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- For the stationary distribution $\phi^{(\infty)}$ we have

$$\phi^{(\infty)} = \phi^{(\infty)} \times P$$

- Whoa! that's just the left eigenvector of the transition matrix!

Power Method

(Horn & Johnson, 1985)

- P has a unique left eigenvector $\phi^{(\infty)}$
  - Called the Perron vector

Power method to compute $\phi^{(\infty)}$

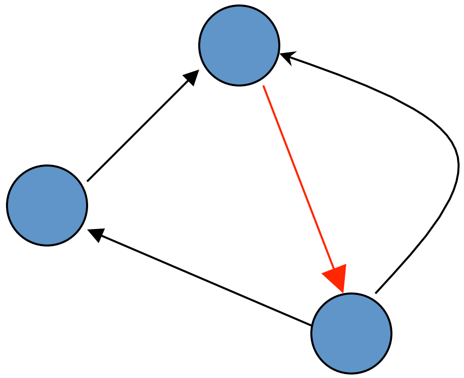
set $\phi^{(0)}$ to be a normalized nonnegative random vector
set $\mathrm{i} = 0$
loop until $\phi^{(0)}, \phi^{(1)}, \ldots, \phi^{(i-1)}, \phi^{(i)}$ converges
    set $\phi^{(i+1)} = P\phi^{(i)}$
    normalize $\phi^{(i+1)}$
    i++
end loop
return $\phi^{(i)}$

Interesting Questions

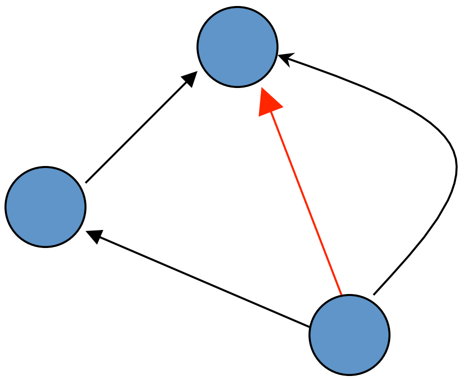
- Does a stationary distribution always exist? Is it unique?
  - Yes, if the graph is “well-behaved”.

Well-behaved graphs

- Irreducible: There is a path from every node to every other node.



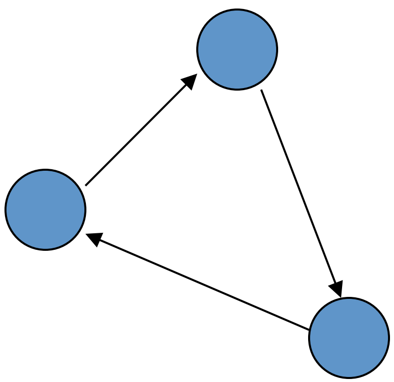
Irreducible



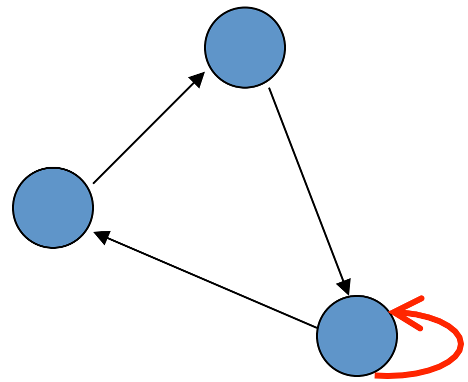
Not irreducible

Well-behaved graphs

- Aperiodic: The GCD of all cycle lengths is *1.* The GCD is also called period.



Periodicity is 3



Aperiodic

Implications of the Perron Frobenius Theorem

- If a Markov chain is irreducible and aperiodic then the largest eigenvalue of the transition matrix will be equal to **1** and all the other eigenvalues will be strictly less than **1**.
  - Let the eigenvalues of P be $\{\sigma_i \mid i=0:n-1\}$ in non-increasing order of σ_i .
  - $\sigma_0 = 1 \mathbf{>} \sigma_1 > \sigma_2 >= \ldots\ldots >= \sigma_n$

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- These results imply that for a well-behaved graph there exists an unique stationary distribution.

Google’s PageRank

- PageRank is a “vote” by all other webpages about the importance of a page
- A link to a page counts as a vote of support
- PageRank uses a random surfer model
  - Occasionally, the surfer gets bored and jumps to a random other page

- “The 25,000,000,000 Eigenvector: the Linear Algebra Behind Google”

Random Walk on Web Graph

- Probability transition matrix given by

$$P_{u,v} = \frac{A_{u,v}}{d_u^{out}} = \frac{A_{u,v}}{\sum_{v=1}^{n} A_{u,v}} \qquad P = D^{-1}A$$

- Use a teleporting random walk (Page et al., 1998) to ensure that the graph is strongly connected and aperiodic:

$$P_{teleport} = \eta P + (1-\eta)\frac{\mathbf{1}\mathbf{1}^T - I}{|V|}$$

PageRank